F-Factors of Graphs:
a generalized Matching Problem

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1. Introduction

In the following we describe a generalization of factors of an undirected graph $X=(V(X),E(X))$ introducing the concept of an F-factor of X. This research has been motivated by a special edge-colouring problem, the latter being a generalization of the well known Magic Squares.

In this paper we shall concentrate on the questions as to which classes of graphs contain an F-factor and how to determine an F-factor of a graph X, if it contains one.

We shall show that in the case of bipartite graphs F-factors are equivalent to the perfect matchings of an graph and that for arbitrary graphs an F-factor is a natural generalization of regular factors as defined by PETERSEN (/7/) and investigated extensively by KÖNIG (/6/).

An algorithmic method for finding an F-factor of an given graph X is presented, which is based on alternating path in a graph. This in turn establishes the connection with matching problems. Based on an algorithm for finding a maximal matching, the algorithm described finds an F-factor of an graph or halts if the graph does not contain such an factor. The algorithm is polynomially time bounded.

2. Definitions

An undirected graph $X=(V(X),E(X))$ consists of a set $V=V(X)$, the vertices, and a set $E=E(X)$ of unordered pairs $e=[x,y]$ of different elements out of V. The set E is the set of edges of X

We shall restrict the investigation to finite graphs and without loss of generality we can assume X to be connected.

Definition 2.1: A matching is defined to be a set **M** of edges, so that no two edges of M are adjacent. A vertex x is said to be saturated by a matching **M** if an edge of **M** is attached to x. A maximal matching is a matching **M** such that the number of edges is maximum. A matching that saturates all vertices of **X** is called a **perfect matching.**

Definition 2.2: A path **W** in **X** is called an alternating path with respect to a matching **M** if the edges of **W** are alternately in M and in E(X)-**M**.

Definition 2.3: An alternating path is called an augmenting path if it connects two unsaturated vertices.

Theorem 2.1: A matching **M** is maximal if, and only if, there exists no augmenting path with respect to **M**. (/1/)

A plain interpretation of the idea of an F-factor illustrates that it is a generalization of a perfect matching and of regular factors, in the following sense: vertices can be saturated by circles also.

One determines a set of vertex disjoint circles $\{K\}$, $V(K^i) \cap V(K^j) = \emptyset$, $i \neq j$, so that each vertex $x \in V(X)$ belongs to exactly one circle. (For the sake of simplicity let us assume for the moment this set of circles to be nonempty.) This set $\{K\}$ of disjoint circles can be partitioned into circles having an even number of edges and into those having an odd number, i.e. $\{K\} = \{K_{2i}\} \cup \{K_{2i+1}\}$, $i=1,2,\ldots$. Trivially, for each circle $K_{2i} \in \{K_{2i}\}$ a perfect matching L_i can be determined, so that each $x \in V(K_{2i})$ is saturated with respect to L_i. Therefore the vertices of X can be saturated either by $\{L_i\}$ or by $\{K_{2i+1}\}$.

In general such a system of circles saturating all elements of V(X) might not exist. Assume now $\{K\}$ saturating a partial set of V(X), i.e. $V(\{K\}) \subseteq V(X)$. If in addition there exists a matching L_0 saturating exactly $V(L_0) = V(X) - V(\{K\})$ then we say that $L_0 \cup \{L_i\} \cup \{K_{2i+1}\}$ spans an F-factor of X. Obviously, both cases $V(\{K\}) = \emptyset$ or $L_0 = \emptyset$ are included and are called perfect matching and regular factor of degree 2 respectively.

Definition 2.4: Let $X(V,E)$ be an undirected graph and $F=(V_F,E_F)$ a spanning subgraph of X, i.e. $V=V_F$ and $E_F \subseteq E$ without isolated vertices. Then F is said to be an **F-factor** of X, if the components of F consist of pairwise nonadjacent edges and/or vertex-disjoint circles each having an odd number of edges.

Example:

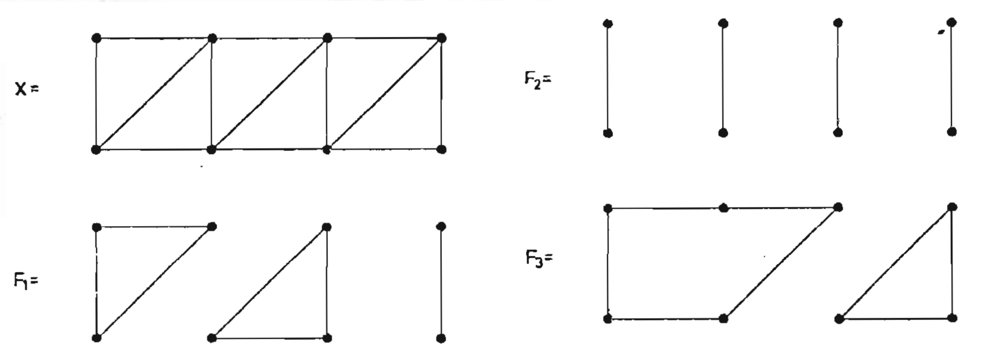


FIG. 2.1

We say that an F-factor decomposes into a linear component L, which is a matching in X, and into a circuit component $\{K_{2i+1}/i=1,2,\dots\}$; the latter consists of pairwise vertex disjoint circuits having $2i+1$ edges each.

Special cases are $L = \emptyset$ or $\{K_{2i+1}\} = \emptyset$. A circuit K_{2i+1} is called an odd circuit.

The following graphical representation of an F-factor might be convenient, assuming L and $\{K_{2i+1}\}$ to be nonempty.

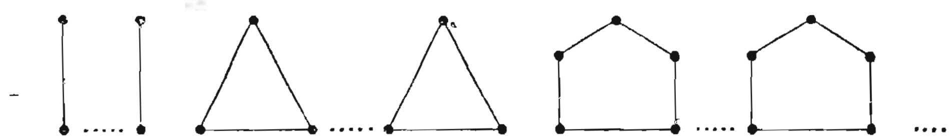
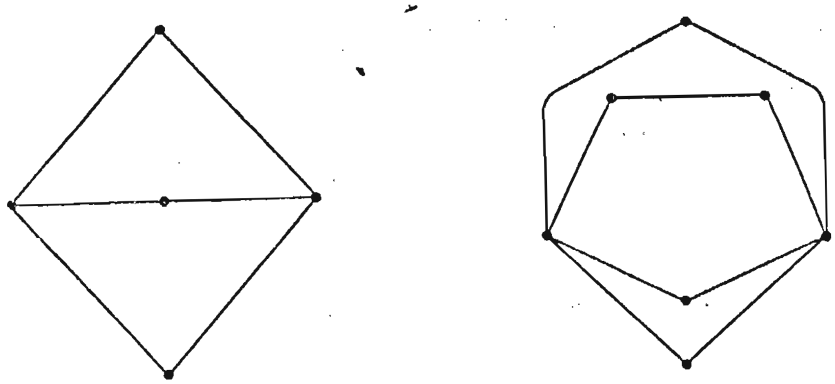


FIG. 2.2

Neither graph of FIG. 2.3 contains any F-factor.



Interesting special cases for F-factors are Hamiltonian Circuits and perfect matching as already noted. If X contains a Hamiltonian Circuit H, then H consists either of an even number of edges or of an odd number of edges. In the first case a perfect matching can be found trivially; in the second case H itself forms an F-factor.

3. Properties of F-factors and related concepts

Earlier results of PETERSEN (/7/) and KÖNIG (/6/) regarding regular factors of degree 1 and of degree 2 can be used for several theorems of existence of F-factors: a regular factor of degree 1 is usually called a perfect matching now, and a regular factor of degree 2 consists of spanning subgraphs whose components are circuits. It should be clear how to construct an F-factor given a regular factor of degree 2.

Theorem 3.1: The complete graph $X = \langle 2n \rangle$ with 2n vertices and n(n-1)/2 edges, n=1,2,...., contains an F-factor with $\{K_{2i+1}\} = \emptyset$.

Theorem 3.2: The complete graph $X = \langle 2n+1 \rangle$ with 2n+1 vertices and n(n+1)/2 edges, n=1,2,...., contains an F-factor with $\{K_{2i+1}\} \neq \emptyset$.

Corollary 3.1.: The complete graph $X = \langle 2n+1 \rangle$, n=1,2,...., contains an F-factor, whose circuit component consists of exactly one triangle.

Proof: given $X = \langle 2n+1 \rangle$ delete one vertex, say a, and all 2n edges adjacent to a. This yields a complete graph $X' = \langle 2n \rangle$. According to theorem 3.1 X' contains an F-factor which is a perfect matching M. Take any edge $e = [u,v]$ out of M and construct a triangle spanned by $T = \{[u,v],[u,a],[v,a]\}$. $M \cup T$ spans the stated F-factor.

Theorem 3.3: A regular graph of even degree, i.e. an Eulerian graph, contains an F-factor.

In the sequel we shall regard the F-factor problem as a generalized matching problem. In particular the algorithms used for finding an F-factor of X are based on the methods of finding alternating paths as is usual in matching algorithms.

Theorem 3.4: A bipartite graph X contains an F-factor, iff X contains a perfect matching.

Proof: Because X is bipartite it does not contain any odd circuit. Therefore $\{K_{2i+1}\} = \emptyset$.

An immediate conclusion of theorem 3.4 is the following:

Theorem 3.5: Every regular bipartite graph X contains an F-factor.

So far as trees are concerned the construction of F-factors is rather simple. Trees are bipartite graphs and according to theorem 3.4 the problem is reduced to a matching problem.

For the F-factor problem (as well as for the matching problem) restriction to graphs having only inner-vertices (i.e. vertices of degree 2) is possible without loss of generality: Let $a \in V(X)$ be an endnode and $e = [a,b]$ the only one incidenting edge. Then the vertex a must be saturated by F, if X contains an F-factor F. So it is sufficient to investigate the graph

$$X_1 = (V_1, E) \text{ with } V_1(X_1) = V(X) - \{a,b\} \quad \text{and}$$
$$E_1(X_1) = E(X) - \{[b,x] / [b,x] \in E(X)\}$$

This reduction can be repeated until the yielded graph contains inner nodes only.

If a graph X contains an F-factor then these F-factors of the set of F-factors $\{F\}$ are distinguished: that whose linear component has a maximal number of edges; and given $|L|$, those which have a maximum number $|K_3|$ of triangles etc. This leads to the following.

Definition 3.1: Let k_{2i+1}, $i=0,1,2,\ldots$ denote the number $|L|$ for $i=0$ and the number of circuits of length $2i+1$, $i=1,2,\ldots$ of a given F-factor F. We call $\langle k_1,k_3,\ldots,k_{2i+1},\ldots,k_{2r+1}\rangle$ with $k_{2r+1}\neq 0$ and $k_{2(r+j)+1}=0$ for $j \geqslant 1$ a characteristic vector of F.

Two characteristic vectors $k^1=\langle k_1^1,k_3^1,\ldots,k_{2s+1}^1\rangle$ and $k^2=\langle k_1^2,k_2^2,\ldots,k_{2r+1}^2\rangle$ can be compared lexicographically. If necessary, the dimension of one vector has to be extended by rightmost zeros before comparing.

$k^1 > k^2$ holds, iff $(k_1^1 > k_1^2)$ v

$(k_i^1 = k_i^2, i=1,2,\ldots,l)$ & $(k_{l+1}^1 > k_{l+1}^2$)).

$k^1 = k^2$ holds, iff $(r=s)$ & $(k_i^1 = k_i^2)$, für $i=1,2,\ldots,s$.

Definition 3.2: Let F_1, F_2 be two F-factors of X and k^1, k^2 their associated characteristic vectors. Then F_1 is said to be greater than F_2, $F_1 > F_2$, iff $k^1 > k^2$ holds.

Remark: One should notice that $k^1 = k^2$ implies F_1 to be isomorphic to F_2, i.e. F_1 and F_2 might be different subgraphs of X.

Definition 3.3: Let be $\{F\}$ the set of F-factors of a graph X. If $\{F\} \neq \emptyset$ then $F_k \in \{F\}$ is called canonical, if $F_k \geq F$ for all $F \in \{F\}$.

The following properties of canonical F-factors are worthwhile to be mentioned: if F_k is a canonical factor of X and if K^1 belongs to the circuit component of F_k, then K^1 does not contain a chord in X, i.e. there is no edge $e = [x,y] \in E(X) - E(K^1)$ with $x,y \in V(K^1)$. Further, two odd circuits K^1, K^2 of a canonical F_k of X cannot be connected by an edge, i.e. there is no edge $e = [x,y] \in E(X)$ with $x \in V(K^1)$ and $y \in V(K^2)$. A proof of the above properties can easily be given by contradiction.

Unfortunately, given an arbitrary F-factor and decomposing it according to the above concept by finding chords and linking edges does not produce a canonical factor in general.

The following theorem shows an immediate connection with matching problems and illustrates also why the concept of canonical F-factors might be central for further research.

Theorem 3.6: Let F_k be a canonical F-factor of X. Then a maximal matching M_F on X can be established from F_k in $O(|V(X)|)$ steps.

Proof: First we present an algorithm which transforms a given canonical F-factor into a maximal matching, then we prove it to work correctly. The time bound $O(|V(X)|)$ is obvious.

Algorithm 3.1: Input: F_k, canonical, respresented by a set of edges spaning F_k in X.

Output: a maximal matching M_F in X.

(i) M_F:=L; /* M is initialized by L of F_k */

(ii) for every circuit K_{2i+1} of F_k

do;

define a maximal matching

M_{2i+1} on K_{2i+1};

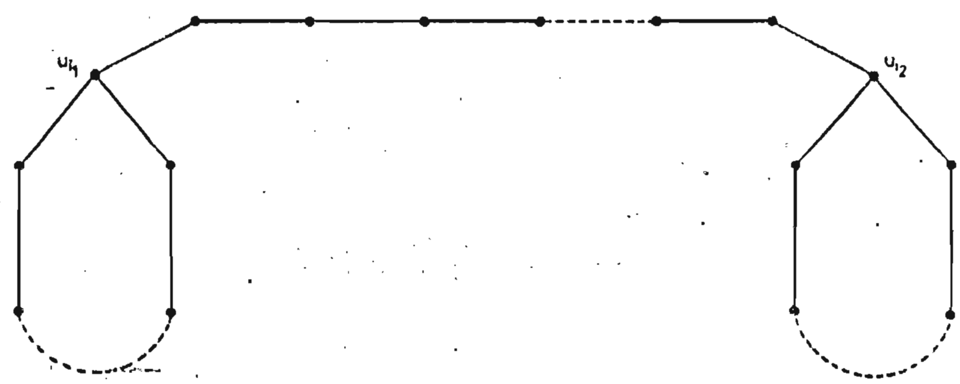
$M_F := M_F \cup M_{2i+1}$;

od;

If F_k just consists of a linear component only all is trivial. Let K_{2i+1} be an odd circuit so that M_{2i+1} consists of i edges. Because the circuits are vertex-disjoint the set M_F is always a matching in X while repeating loop (ii).

The number r of vertices not saturated by M_F equals the number of odd circuits of F_k and, evidently, M_F is a maximal matching for r=1.

Let $U=\{u_1,u_2,\dots,u_r\}$ be the set of unsaturated vertices, $r \geq 2$. If M_F was not maximal an augmenting path $W(u_{i_1},u_{i_2})$ must exist connecting two odd circuits K_{2i+1}, K_{2j+1}, i,j not necessarily distinct.



Therefore $M_1 = (M_F \cup (E(W)) - (M_F \cap E(W))$ would be a matching with $|M_1| = |M_F| + 1$ and the vertices u_{i_1}, u_{i_2} would be saturated by edges of M_1. In addition the linear component L of F_k could be enlarged by $M_1 \cap E(K_{2i+1})$ and by $M_1 \cap E(K_{2j+1})$ which is a contradiction to F_k being canonical.

Theorem 3.7: Given a maximal matching M and a canonical factor F_k of X. If $U = \{u_1, u_2, \ldots, u_r\}$ is the set of unsaturated vertices with respect to M, then F_k contains exactly $r = |U| = |\{K_{2i+1}\}|$ odd circuits.

The theorem 3.7 is an evident corollary of theorem 3.6.

4. Constructing F-factors

In this chapter we are interested in finding an F-factor of a given graph X or in deciding whether X contains one. We shall use a method which is based on alternating paths (/2/, /3/, /4/). This underlines the immediate implication to matchings once more. Without loss of generality X is assumed to be connected and to have innernodes only. The latter is not relevant in the sequel and rather should be seen as an argument for reducing the average number of iterations.

We start with an algorithm which eventually finds an F-factor. The aligned discussions of why this method could fail are useful for the algorithm 4.2 presented finally in this paper.

Algorithm 4.1: Input: $X = (V,E)$, connected,

Output: an F-factor $F=(V,E(F))$

in the case $P = \emptyset$, as defined below.

```
S0:  E(F)=∅; P=∅     /* initializing the sets E(F) and P */
S1:  construct a maximal matching M;
     E(F):=M;
S2:  /* Let be U = {u_1,...,u_r} the set of unsaturated
        vertices */
     while U = ∅ do;
      S2.1:  take u_i ∈ U, generate a vertex ū_i ∉ V(X)
             and construct the following graph X̄:
             V(X̄):=V(X) ∪ {ū_i};
             E(X̄):=E(X) ∪ {[ū_i,x]/ [u_i,x] ∈ E(X)}
      S2.2:  Try to find an augmenting path W(u_i,ū_i)
             in X̄ with respect to M.
             if ∄ W(u_i,ū_i)
                then U:=U-{u_i} ; P:=P ∪ {u_i};
                else do;
                        /* let W=[u_i,x_1] ,[x_1,x_2],....
                           ...,[x_{s-1},x_s] ,[x_s,ū_i]
                           be the found augmenting path with
                           {[x_1,x_2],[x_3,x_4] , ....
                           ...,[x_{s-1},x_s]} ⊂ M
                           {[u_i,x_1] ,[x_2,x_3], ...
                           ...,[x_s,ū_i]} ⊂ E(X̄) - M */
                           construct W̄ by
                           E(W̄):= (E(W) ∪ [x_s,u_i]) -
                           [x_s, ū_i];
                  od;
```

S2.3: /* $\bar{W}$ is an odd circuit K_{s+1} with

$[u_i,x_1],[x_1,x_2],\ldots,[x_s,u_i]$ */

$E(F):=E(F) \cup E(K_{s+1})$; /* $E(F):=E(F) \cup \bar{W}$ */;

$U:=U-\{u_i\}$

od;

S3: If $P = \emptyset$ then $E(F)$ spans a F-factor of X;

Establishing an augmenting path $W(u_i,y)$ the case $y \in U$ is not possible because otherwise M would not be maximal. Therefore $y = \bar{u}_i$ if such a path $W(u_i,y)$ exists. An odd circuit is obtained by an identification of $\bar{u}_i$ with u_i. This circuit is added to $E(F)$. In addition, while construction augmenting paths $W(u_j,\bar{u}_j)$ one knows also that such a path $W(u_j,\bar{u}_j)$ does not reach any vertex of a circuit added to F already. If this was possible an augmenting path $W(u_j,u_i)$ would be constructable in contradiction to the maximality of M.

On the other hand, as shown in FIG. 4.1, algorithm 4.1 does not find necessarily an augmenting path $W(u_i,\bar{u}_i)$ in $\bar{X}$ with given matching M though X is assumed to have an F-factor.

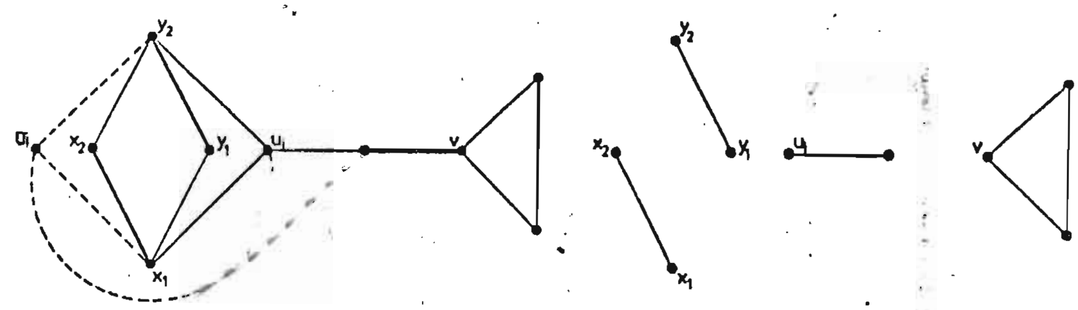


FIG. 4.1

If X contains an F-factor, then a canonical F-factor F_k exists naturally. This factor F_k induces a maximal matching M_F as shown in theorem 3.6. This matching M_F leaves the same number of vertices unsaturated as the number of odd circuits of F_k. According to the construction rules of M_F it follows that every unsaturated vertex is contained in exactly one circuit of F_k. This implies $M_F \neq M$ so far as the matching M of algorithm 4.1 is concerned for the case that this algorithm does not find an augmenting path.

The figure 4.2 below illustrates this chain of arguments.

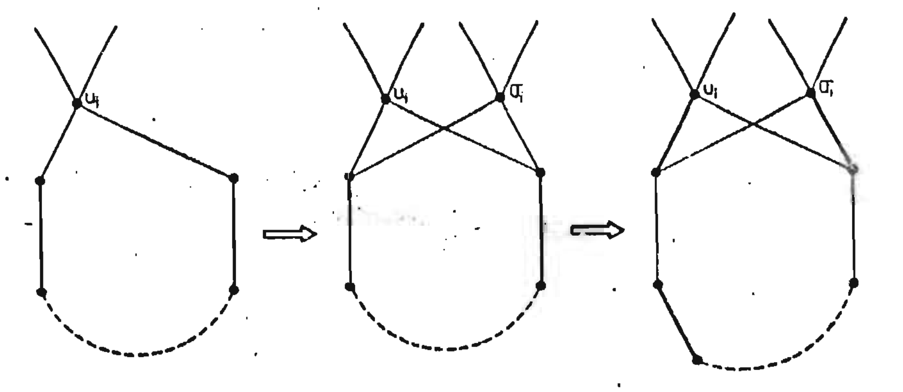


FIG. 4.2

C. BERGE (/1/) has shown the following lemma:

Lemma 4.1: Let M be a maximal matching and W either an alternating path or an alternating circuit in X consisting of "dark" and "light" edges alternately. Let the operation which interchanges dark (i.e. $e \in M$) and light edges (i.e. $e \in E(X)-M$) in W be called a "transfer" on W. Each maximum matching $\bar{M}$ can be obtained from M by transfers along alternating circuits or alternating paths which start at an unsaturated vertex.

Applying lemma 4.1 if algorithm 4.1 terminates unsuccsessfully we obtain the following: the set P contains those unsaturated vertices $u_i \in U$ which have not yet been saturated by one of the circuits constructed according step S2.3. By virtue of lemma 4.1 one has to find the set $\{AW(p,x)\}$ of alternating paths $AW(p,x)$, for every $p \in P$. Take any $AW(p,y)$ so that y can be saturated by an odd circuit as stated in steps S2.1 - S2.3. This is possible iff X contains an F-factor. No path $AW(p,x)$ can reach a vertex x which has been saturated previously by an odd circuit, because this would form an augmenting path which would be contradiction to M being maximal.

So we can formulate the following algorithm for finding an F-factor in a graph X.

Algorithm 4.2: input: $X = (V,E)$, connected

output: either an F-factor of X or the message "X does not contain an F-factor".

```
T1:  apply algorithm 4.1;
T2:  while P ≠ Ø do;
     T2.1:  choose p ∈ P;
     T2.2:  construct the set of alternating paths
            AW(p,x) =def AW
            /* this step is bounded by |V(X)|^3 as it is
               shown in (GA/76/) */
     T2.3:  NOTFOUND:= true;
            while NOTFOUND & (AW ≠ Ø);
               do;
                 choose a path AW(p,x); delete it; i.e.:
                 AW: = AW - AW(p,x);
                 apply steps S.2.1, S2.2 of algorithm 4.1
                 accordingly for finding an augmenting path
                 W(x,x̄) given the matching
                 (M ∪ AW(p,x)) - (M ∩ AW(p,x)).
                 If one has found such a path W(x,x̄)
                    then NOTFOUND:= false;
               od;

     T2.4: if NOTFOUND then do;
                                  print "X does not contain a
                                  F-factor";
                                  stop
                               od;
```

T2.5: M: = (M $\cup$ AW(p,x)) - (M $\cap$ AW(p,x));

/* note: p is saturated by M, but x is not */

P: = P - {p}

T2.6: /* Apply step S2.3 accordingly */

Construct the odd circuit K_{s+1} by

$[x,x_1],[x_1,x_2],\ldots,[x_s,\bar{x}]$

exchanging $[x_s,\bar{x}]$ by $[x_s,x]$;

E(F): = E(F) $\cup$ E(K_{s+1});

E(F): = (E(F) $\cup$ AW(p,x)) - (E(F) $\cap$ AW(p,x);

od;

T3: E(F) spans an F-factor of X.

5. Further research

First of all a highly efficient implementation of an algorithm finding an F-factor should be of interest. The particular question is wheter such an algorithm has the same time complexity as a matching algorithm (see /4/) has in the best case. For the case of a bipartite graph this obviously holds.

So far as theorem 3.6 is concerned an algorithm for finding a canonical F-factor might be central for further investigations. We conjecture that the fundamental system of circuits plays an important role for establishing such an algorithm. If this were so a different approach to matching problems would have been found, not based only on alternating path methods as current algorithms are.

6. Acknowledgment

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