Canonical F-Factors of Graphs

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0. Summary:

In this paper we are carrying on the work about F-factors which have been introduced by Mühlbacher. Beside further general structural statements we declare L-canonical factors F_L and C-canonical factors F_C in addition to the earlier defined canonical factor F_K. $\{F_K\} \subseteq \{F_L\} \subseteq \{F_C\}$ holds. An algorithm, transforming every factor from one of these classes into a maximal matching in C(n) steps, is specified. In addition we can determine in $O(n^3)$ steps whether a factor is a C-canonical one or not.

1. Introduction

In this paper we carry on the studies started in /MÜ79b/. Their origin is a problem in the theory of numbers /MÜ79a/. For reasons of completeness essential definitions and results of /MÜ79/ are repeated and further properties are deduced. We keep suggesting undirected graphs X without multiple edges and loops.

Definition 1.1:

A spanning subgraph F of an undirected Graph X=(V,E) is called F-factor of X, iff the components of F are non adjacent edges and/or disjoint circles each having an odd number of edges.

Example:

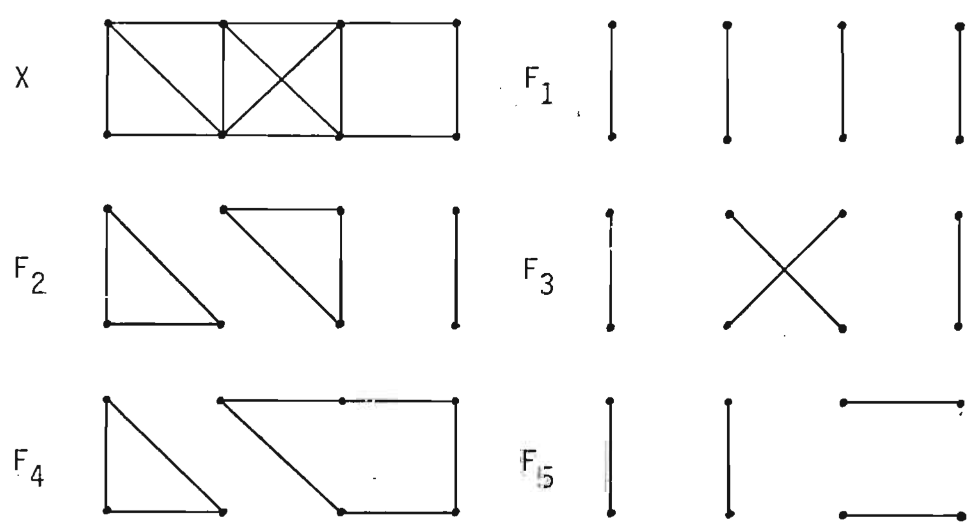

Fig. 1.1

Figure 1.1 shows a graph X and some, but not all of its F-factors. There also exist graphs containing no F-factor. The F-factor F_1 in the preceding example shows the close connexion to the matchingproblem: F-factors may be interpreted as a generalization of maximal matchings.

Definition 1.2:

Let f_{2i+1}, $i = 0,1,2,\ldots$ for $i=0$ be the number $|L|$ of the nonadjacent edges of a F-factor F and for $i \geq 1$ the number of circles with lengths $2i+1$ of F. We call $\vec{f} = \langle f_1,0,f_3,0,f_5,0,\ldots,\ldots f_{2i+1},0,\ldots,f_{2r+1}\rangle$ with $f_{2r+1} \neq 0$ and $f_{2(r+j)+1} = 0$ for $j \geq 1$ the characteristic vector $\vec{f}$ of F.

Definition 1.3:

Let F_1 and F_2 be two F-factors of a graph X with characteristic vectors $\vec{f}_1$ and $\vec{f}_2$, $F_1 \geq F_2$ holds, iff $\vec{f}_1 \geq \vec{f}_2$ holds lexicographically.

Remark:

In originally $\vec{f}$ is defined to be $\langle f_1,\ldots,f_{2i+1},\ldots\rangle$. For formal reasons we prefer the notation above, because in this case the indices of f_i correspond with usual notation of vectors.

Definition 1.4:

Let $\{F\}$ be the set of F-factors of a given graph X and $\{F\} \neq \emptyset$, we call $F_K \in \{F\}$ canonical, iff $F_K \geq F$ for all $F \in \{F\}$ holds. For canonical F-factors we will use the symbol F_K.

In /MÜ79b/ following theorems were proofed.

Theorem 1.1:

Let F_K be a canonical factor of X. It is possible to generate a maximal matching M_F of X in $O(|V(X)|)$ steps.

Theorem 1.2:

Given: a maximal matching M and a canonical factor F_K of X. If $U = \{u_1,u_2,\ldots,u_r\}$ is the set of unsaturated vertices with respect to M, then F_K contains exactly $r = |U|$ odd circles.

Now the question how to get algorithmically a canonical factor of a given graph X arises. We have not solved this problem and so it seems to be interesting to establish structural properties of F-factors generally and F_K factors specially.

2. Some numerical properties of F resp. F_K factors

Lemma 2.1:

Let F be a F-factor of X and $\vec{f}$ the characteristic vector of F $<f_1,0,f_3,0,f_5,\dots f_{2r+1}>$, then:

$$n = |V(X)| = f_1 + \sum_{j=0}^{r} (2j+1) * f_{2j+1} = f_1 + \sum_{j=1}^{2r+1} jf_j \quad \text{holds.}$$

Proof:

trivial for edges resp. circles are disjoint.

Lemma 2.2:

Let F and $\vec{f}$ defined as before then

$$|E(F)| = \sum_{j=0}^{r} (2j+1) * f_{2j+1} = \sum_{j=1}^{2r+1} jf_j \quad \text{holds.}$$

Proof:

trivially, note, that for even j the value f_j is zero.

Corollary 2.1:

$|E(F)| = n-f_1$

Theorem 2.1:

Let X be a graph and F a F-factor of X then:
the number of edges (nodes) which are contained in circles of F is $2|E(F)| - n$

Proof:

Using lemma 2.1 and 2.2 we get:

$$2|E(F)| = 2 * \sum_{j=1}^{2r+1} jf_j = \sum_{j=1}^{2r+1} jf_j + f_1 + \sum_{j=2}^{2r+1} jf_j = n + \sum_{j=2}^{2r+1} jf_j.$$

The last summation is exactly the number of edges (nodes) which are contained in circles of F.

Theorem 2.2:

Let X be a graph and F a F-factor of it then:
$1/2\ n \leq |E(F)| \leq n$ holds.

Proof:

The left inequality holds trivially. Let $d(x)$ be the degree of $x \in V(X)$ then concerning F: $1 \leq d(x) \leq 2$ holds. Consequently

$$|E(F)| = \frac{1}{2} \sum_{x \in V(F)} d(x) \leq \frac{1}{2} \sum_{x \in V(X)} 2 \leq n.$$

The following theorem shows canonical F-factors meeting a property of minimality related to their set of edges.

Theorem 2.3:

Let X be a graph and $\{F^i \,/\, i = 1,2,...s\}$ the set of F-factors of it and F_K be a canonical F-factor of it. Then

$$|E(F_K)| = \min_{1 \leq i \leq s} \{|E(F^i)|\}$$

Proof:

For F_K $f_1 \underset{def}{=} {}_k f_1$ is maximum per definition. Therefore

$|E(F_K)| = n - \max_{1 \leq i \leq s} {}_i f_1$ holds. For n is constant

$|E(F_K)| = \min_{1 \leq i \leq s} (n - {}_i f_1) = \min_{1 \leq i \leq s} (|E(F^i)|)$ holds consequently

according corollary 2.1.

3. Structural properties of F- resp. F_K-factors

Union and intersection of graphs are defined as usually /BE 73/, /DOMÜ73/. Analogously we introduce the difference of graphs:

Definition 3.1:

Let X_i and X_j be graphs, we define the difference $(X_i - X_j)$ as follows:

$V(X_i - X_j) = V(X_i) - V(X_j)$

$E(X_i - X_j) = \{[x,y] \mid x,y \in V(X_i - X_j),\ [x,y] \in E(X_i)\}$

In the case that $V(X_j) \subset V(X_i)$, getting the difference of X_j and X_i corresponds to the removal of the nodes $x \in V(X_j)$ from $V(X_i)$ and the accompanying deletion of edges which have become impossible now.

Theorem 3.1:

Let X be a graph and F a F-factor of X and S any component of F. Then F-S is a F-factor of X-S.

Proof:

For F is a F-factor, F is a spanning subgraph of X that means V(F) = V(X). Consequently V(F-S) = V(X-S) and therefore F-S is F-factor of X-S.

The following theorem gives us the formal requirements for an eventual use of a divide and conquer algorithm.

Theorem 3.2:

Let Y,Z be subgraphs of X so that V(Z) = V(X)-V(Y) and let F_Y resp. F_Z be F-factors of Y resp. Z, then $F_Y \cup F_Z$ is a F-factor of X.

Proof:

$V(Z) \cap V(Y) = \emptyset$. Consequently $F_Y \cup F_Z$ contains only nonadjacent edges and odd circles as components. For $V(Z) \cup V(Y) = V(X)$ $V(F_Y \cup F_Z) = V(X)$ holds. Consequently $F_Y \cup F_Z$ is a F-factor of X.

Theorem 3.3:

Let F be a F-factor of a graph X and $Z \subset X$. Follows from $[u,v] \in E(F)$, either $u,v \in V(Z)$ or $\{u,v\} \cap V(Z) = \emptyset$ then F-Z is a F-factor of X-Z.

Proof:

Let [u,v] be an edge of the linearpart of F. In this case this edge is contained in E(F-Z) without any adjacent edges or isn't contained in E(F-Z) at all.

Let [u,v] be an edge from a circle K of F. For the preceding conditions K is either disjoint to E(F-Z) or K is contained in F-Z completely. For V(F)=V(X) F-Z is consequently a F-factor of X-Z.

For completeness we mention the following theorem which allows the restriction of our studies on connected graphs without loss of generality:

Theorem 3.4:

A graph X with n components contains a F-factor exactly, iff every component of X contains a F-factor.

Theorem 3.5:

If F_K is a canonical F-factor of a graph X, then

$$X - L(F_K) = K(F_K).$$

Remove the linearpart of F_K and the incident edges, so the set of disjoint circles, the so-called circlecomponent $K(F_K)$, of the factor F_K remains exactly.

Proof:

Assume that $X - L(F_K)$ contains edges $[u,v]$ with $[u,v] \notin E(K(F_K))$ and let $[u,v]$ be such an edge.

(i) if u,v are nodes contained in the same circle K_{2i+1}, then $[u,v]$ is a chord of the circle K_{2i+1}. In this case K_{2i+1} can be disassembled into a "linear part" and a less odd circle K_{2j+1}, $j < i$. This is a contradiction to F_K being canonical.

(ii) let u,v be contained in disjoint circles K^1, K^2. Then $[u,v]$ is an odd path between K^1 and K^2 and we can disassemble K^1, K^2, $[u,v]$ and $L(F_K)$ becomes larger. This is a contradiction to F_K being canonical.

$X - L(F_K)$ contains no isolated nodes, therefore the theorem holds.

4. Maximizing the linear part

An essential property of canonical F-factors is the fact, that their linearpart is a maximum. This property is used for proofing the fact, that we can establish a maximal matching in $O(|V(X)|)$ steps. We are showing now, that the maximality of the linearpart is a sufficient condition for establishing a maximal matching in $O(V|(X)|)$ steps.

Definition 4.1:

Let X be a graph and F a F-factor of X. We call the factor F L-canonical, iff its linearpart is a maximum, i.e, iff for all F-factors F^i $|E(L(F^i))| \leq |E(L(F))|$ holds.

Obviously every canonical factor is L-canonical.

Theorem 4.1:

Let F_L^1, F_L^2 be L-canonical factors of a graph X, then $|E(F_L^1)| = |E(F_L^2)|$.

Proof:

From $|E(L(F^1))| = |E(L(F^2))|$ follows $|V(L(F^1))| = |V(L(F^2))|$ consequently for nodes contained in circles $|V(K(F^1))| = |V(K(F^2))|$ and therefore $|E(K(F^1))| = |E(K(F^2))|$ holds.
From these equations the thecrem follows obviously.

Theorem 4.2

Let F_L be a L-canonical factcr of a graph X. Using the following algorithm one can establish a maximal matching M_F of X in $O(|V(X)|)$ steps.
Input: F_L L-canonical with linearpart $L(F_L)$
Output: maximal matching M_F.

```
 (i)  /* Initialisation */
      M_F := L(F_L);
(ii)  For every circle K_{2i+1} in F_L
      do;
         establish a maximal matching M_{2i+1} of K_{2i+1}
         M_F := M_F ∪ M_{2i+1};
      od;
```

Proof:

The set of edges, established by this algorithm, is obviously a matching of X. Assume M_F not maximal and let $U = \{u_1,u_2,\ldots,u_n\}$, $r \geq 2$ be the set of unsaturated nodes concerning M_F. Then, there is an augmenting path $W(u_i,u_j)$ connecting two unsaturated nodes. Establishing M_{2i+1} the choice of the unsaturated node u_i in K_{2i+1} is optional. Therefore u_i,u_j can be chosen, that the augmenting path $W(u_i,u_j)$ connects two nodedisjoint circles K_{2i+1}, K_{2j+1} and u_i is the first node of K_{2i+1} met by this path and u_j is the first of K_{2j+1}. (Fig. 4.1).

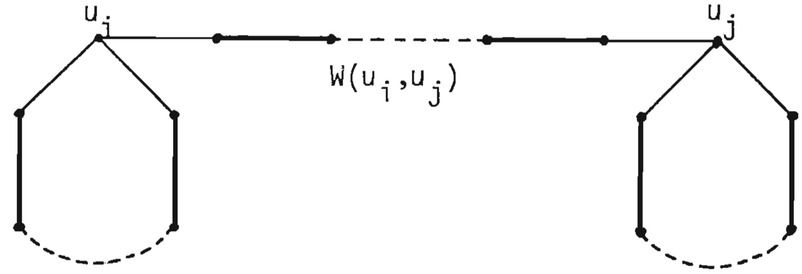

Fig. 4.1

In this case u_i, u_j can be saturated by substituting $E(W(u_i,u_j))-(E(W(u_i,u_j)) \cap M_F)$ for $E(W(u_i,u_j) \cap M_F)$. This substitution contradicts the maximality of $|E(L(F))|$ and consequently F being L-canonical. The bound $O(|V(X)|)$ is evident.

An obvious consequence of theorem 4.2 is the following:

Theorem 4.3:

Let F_L^1, F_L^2 be two L-canonical factors of X, then the number of circles contained in F_L^1 is equal to the number of circles contained in F_L^2.

Proof:

From the proof of theorem 4.2 follows, that the number r of nodes, unsaturated related to a maximal matching is equal to the number of odd circles contained in a L-canonical F-factor. Consequently the number of unsaturated nodes relative to the matching established by F_L^1 is exactly equal to the number of unsaturated nodes related to the matching established by F_L^2. Therefore the theorem holds.

Theorem 3.5 can be transfered to L-canonical factors because proofing the theorem only the property $|E(L(F_K)|$ being maximal has been used:

Theorem 4.4:

Let F_L be a L-canonical factor of X, then
$X - L(F_L) = K(F_L)$ holds.

5. Minimizing the circle part

In this section, we will show, that concerning to the construction of a maximal matching using a F-factor F, the minimality of the circle part of the factor is sufficient: using F-factors with the minimal number of circles contained in it, a maximal matching is established by the algorithm of theorem 4.2 algorithmically, analogous as it has been established using F_L-factors and F_K-factors.

Definition 5.1:

Let X be a graph and F_C a F-factor of X. The factor F_C is called C-canonical, iff the number $|K(F_C)|$ of circles contained in it is minimal, i.e. iff $|K(F_C)| \leq |K(F)|$ holds for all F-factors F of X.

Theorem 5.1:

Let F_C be a C-canonical factor of X, then we can establish a maximal matching M_F in $O(|V(X)|)$ steps starting with F_C.

Proof:

The proof is nearly identical with the proof of theorem 4.2: Assume the established matching not being maximal, then we can reduce the number of circles contained in the F-factor using the augmenting path existing concerning M_C.

This theorem can be inversed:

Theorem 5.2:

Let F be a F-factor of a graph X and establishes the algorithm given in theorem 4.2,used on F, a maximal matching M_C, then F is a F_C-factor, i.e. C-canonical.

Proof:

An obvious consequence of 5.1 is the fact, that the number of unsaturated nodes relative to the established matching is exactly the number of circles contained in F. From the maximality of matching M_C follows that no F-factor may contain less circles than the given one. Otherwise using the algorithm, we could establish a matching, saturating more nodes, and this would be a contradiction to M_C being maximal.

Corollary:

Because we can establish a maximal matching of a graph in $O(n^3)$ steps and our algorithm for establishing M_C is a $O(n)$ algorithm we can decide in $O(n^3)$ steps whether a F-factor F is a F_C-factor or not.

In fig. 5.1 a simple example shows, that theorem 5.2 holds only for F_C factors but not for F_K-factors globaly.

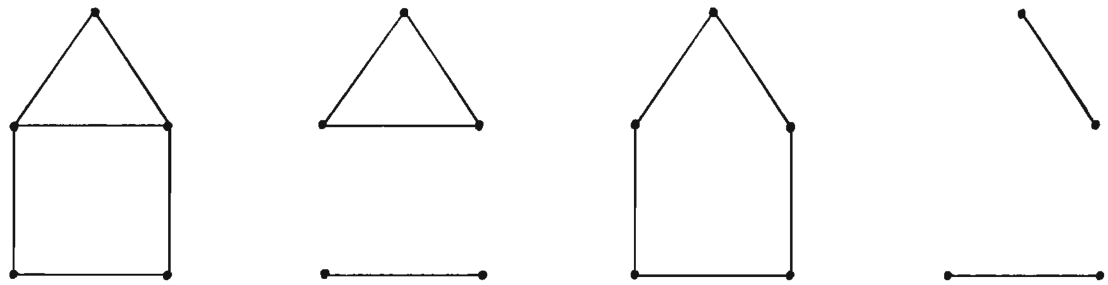

Fig. 5.1

The concept of alternating paths defined for matchings M (note: $M \subsetneq E(X)$) can be transfered to F-factors, substituting $E(F) \subsetneq E(X)$ for M. A consequence of this concept and theorem 5.2 is:

Theorem 5.3:

A F-factor F of a graph X is C-canonical, iff it doesn't contain two circles K^1, K^2 connected by an alternating path related to F.

Proof:

Assume F being a C-canonical factor F_C containing two circles K^1, K^2 as stated. Related to the matching M_C established by the algorithm given in theorem 4.2 two nodes $u_1 \in V(K^1)$ and $u_2 \in V(K^2)$ are unsaturated. If there is an alternating path from K^1 to K^2, then this path is an augmenting path related to M_C. M_C couldn't be maximal. Contradiction!

Conversly let F be a F-factor not containing two circles K^1 and K^2 connected by an alternating path concerning F; assume F not being F_C factor. Then F must contain at least two circles K^1, K^2. Establishing a matching as usually two nodes $u_1 \in V(K^1)$ and $u_2 \in V(K^2)$ are unsaturated related to this matching. Existing no alternating path concerning F, M_F must be maximal because K^1, K^2 are chosen arbitrarily in the set of circles contained in F. From theorem 5.2 follows that F must be F_C-factor.

6. The classes F_K, F_L and F_C

In section 2 and 3 canonical F-factors $\{F_K\}$ have been defined, in section 4 the class $\{F_L\}$ has been defined and in section 5 the class $\{F_C\}$. From definition it follows $\{F_K\} \subseteq \{F_L\}$. The simple example in figure 5.1 shows, that there are F_C factors which are not L-canonical. According to theorem 4.2 and theorem 5.2 every L-canonical factor is a C-canonical one. Therefore the following relation holds:

$$\{F_K\} \subseteq \{F_L\} \subseteq \{F_C\}.$$

However we conjecture that $\{F_L\} \subseteq \{F_K\}$ and consequently $\{F_L\} = \{F_K\}$.

7. Literature:

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