

ON CERTAIN CLASSES OF FRACTIONAL MATCHINGS

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An f -matching in an undirected graph X is defined as a set of vertex disjoint edges and odd cycles. In particular we consider f -matchings which saturate the maximum possible number of vertices and contain a maximum number of vertex disjoint edges. The main result is, that in this case different possible f -matchings in X with these properties contain the same number of triangles, pentagons and so on. This means that maximizing the set of vertex disjoint edges in the f -matchings determines the number of cycles of length 3 (i.e. triangles), 5, ..., $(2n+1)$.

The problem is stated as a linear programming problem called fractional matching problem in a graph X .

1. Introduction

In this paper some research on f -matchings is continued which has been initiated by Mühlbacher in [3] and [4]. We consider undirected graphs $X = (V(X), E(X))$ without loops and without parallel edges. As usual $X \cup Y$ and $X \cap Y$ denote union and intersection of graphs X and Y . The difference $X - Y$ of two graphs X and Y consists of the edges $E(X) - E(Y)$ together with all vertices incident with these edges. The number of components of a graph X is denoted by $|X|$. We write $Y \subset X$ if Y is a subgraph of X and $u \in X$ or $e \in X$ if u is a vertex of X or e is an edge of X .

Let X be a graph with $|V(X)| = n$ and $|E(X)| = m$, let A be the vertex-edge incidence matrix and $\mathbf{1}_n$ the n -vector of 1's. Consider the linear programming problem

$$\begin{aligned} \text{(FMP)} \quad & \text{Maximize } v(z) = \sum_{e \in E(X)} z_e, \\ & \text{subject to } Az \leq \mathbf{1}_n, \quad z \geq 0. \end{aligned}$$

The problem (FMP) is called the *fractional matching problem* in X . For any feasible solution z to (FMP) let $F(z)$ be that subgraph of X which is generated by the edge set $\{e \mid z_e > 0\}$. Clearly $F(z)$ is a matching in X if z is a binary vector. In this case we call $F(z)$ an integer matching. Furthermore, it is known that any basic feasible solution to (FMP) has components equal to 0, 1 or $1/2$ (Balinski [2], Nemhauser

and Trotter [6]). Thus, if z is basic, then the components of $F(z)$ are single non adjacent edges and/or cycles of odd length.

A subgraph $F(z)$ of X where z is feasible is called a *fractional matching* of X , or simply an *f-matching*.

Let z be a basic solution to (FMP) and $F = F(z)$. The subgraph of X consisting of all single non adjacent edges of F is called the *linear part* of F and will be denoted by $L(F)$. Of course, $L(F)$ is an integer matching of X . The union of all cycles of F forms another subgraph of X which is called the *cycle part* of F and is denoted by $C(F)$.

An edge e *saturates* the vertex u (with respect to an f-matching F), if e belongs to $E(F)$ and e and u are incident. Analogously, a cycle C saturates the vertex u , if C is a component of F and contains u .

A basic f-matching $F(z)$ is called a *perfect* f-matching, or simply an *F-factor* of X , if $Az = 1_n$, i.e. if all vertices of X are saturated by components of $F(z)$. Clearly, if $F(z)$ is an F-factor, then z is a maximum basic solution to (FMP), and if X possesses an F-factor, then $F(z')$ is an F-factor for any maximum basic solution z' .

It should be mentioned that the problem of finding a maximum f-matching in a graph X is equivalent to the problem of finding a maximum matching in a bipartite graph X_B derived from X by a transformation technique introduced by Edmonds and Pulleyblank (see Nemhauser and Trotter [6] or Balas [1]). Thus, an F-factor in X corresponds to a perfect integer matching in X_B . (Since X_B is bipartite, all basic f-matchings are integer.)

The existence question was settled by an important theorem of Tutte [8]. In this paper we investigate two particular classes of F-factors and demonstrate that they are identical. The result is used to show that the same identity holds for the analogously defined classes of maximum basic f-matchings.

If X possesses different basic f-matchings $F(z)$ with maximum value $v(z)$, then there are such f-matchings the linear part of which is maximal and others the cycle part of which contains a minimal number of cycles.

Definition 1.1. Let \mathcal{F} be the class of all basic f-matchings $F(z)$ with maximum value $v(z)$. An f-matching $F_L \in \mathcal{F}$ is called *L-canonical*, if the linear part of F_L is maximal in \mathcal{F} , i.e. if

$$|L(F_L)| = \text{Max}_{F \in \mathcal{F}} |L(F)|.$$

An f-matching $F_C \in \mathcal{F}$ is called *C-canonical*, if the number of its cycles is minimal in \mathcal{F} , i.e. if

$$|C(F_C)| = \text{Min}_{F \in \mathcal{F}} |C(F)|.$$

The set of components of a maximum basic f-matching F may be subdivided into classes smaller than $L(F)$ and $C(F)$ by considering the length of the different cycles in F .

Definition 1.2. Let $F \in \mathcal{F}$ and let $i \geq 1$. By $C(F, i)$ we denote that subgraph of F which contains just all cycles of length $2i + 1$.

Of course, $C(F) = \bigcup_{i \geq 1} C(F, i)$. For convenience we put $C(F, 0) = L(F)$.

Now if X contains different maximum basic f -matchings, than these f -matchings can be distinguished by the number of single edges, triangles, pentagons, etc. which they contain. This leads us to the following definition.

Definition 1.3. For $F \in \mathcal{F}$ let $2r + 1$ be the number of edges in the largest cycle of $C(F)$ and let

$$f_i = |C(F, i)|, \quad 0 \leq i \leq r.$$

Then $f = (f_0, \dots, f_r)$ is called the *characteristic vector* of F . The characteristic vectors of the different f -matchings in \mathcal{F} can be ordered lexicographically (where the different lengths of vectors is of no concern).

Definition 1.4. Let F^1 and F^2 be elements of \mathcal{F} with characteristic vectors f^1 and f^2 respectively. We write $F^1 \geq F^2$ if and only if $f^1 \geq f^2$ holds in the sense of the lexicographic order on the set of characteristic vectors. An element $F_K \in \mathcal{F}$ is called *K-canonical*, if $F_K \geq F$ for all $F \in \mathcal{F}$, i.e. if F_K is a maximum with respect to the above introduced order on \mathcal{F} .

For $U \in \{K, L, C\}$ let \mathcal{F}_U be the class of all U -canonical f -matchings of X . C -canonical f -matchings have been considered earlier in the literature, particularly by Uhry in [7] and by Balas in [1]. These authors dealt with the construction of an element $F_C \in \mathcal{F}_C$ and showed that from such an element F_C a maximum integer matching in X is derivable in $O(n)$ steps.

This latter result is based on the following theorem.

Theorem 1.5. Let F be a maximum basic f -matching in X with cycle part $C(F) = C_1 \cup C_2 \cup \dots \cup C_s$, where each C_i is an odd cycle of length $2k_i + 1$. Let M_i be a maximum integer matching in C_i , $1 \leq i \leq s$, and define

$$M(F) = L(F) \cup M_1 \cup M_2 \cup \dots \cup M_s \quad (1)$$

Then $M(F)$ is a maximum integer matching in X if and only if $F \in \mathcal{F}_C$.

The 'if' part of the theorem was proved by Uhry [7] and was rediscovered by Balas in [1] and by Mühlbacher and Steinparz in [4]. The 'only if' part is trivial, since the number s in (1) equals the number of vertices which are left unsaturated by $M(F)$ and this number is the same for all maximal integer matchings. In [4] it was shown that

$$\mathcal{F}_K \subset \mathcal{F}_L \subset \mathcal{F}_C. \quad (2)$$

These inclusions imply the following corollary to Theorem 1.5:

Corollary 1.6. *Let F be any canonical f -matching of no matter which kind. Then $M(F)$ as defined in (1) is a maximum integer matching in X .*

There are graphs X for which $\mathcal{F}_L \neq \mathcal{F}_C$ (see Fig. 1). For the other inclusion in (2) it was conjectured in [4] that $\mathcal{F}_L = \mathcal{F}_K$. The rest of the paper is devoted to the proof of this identity which implies that if F is a maximal f -matching of X for which the number of edges in cycles is minimized then its characteristic vector as defined in 1.4 is lexicographically maximal over all such characteristic vectors.

2. K-canonical and L-canonical f -matchings

We claim that the following statement is true:

Theorem 2.1 *For any graph X the class \mathcal{F}_L of all L-canonical f -matchings is identical with the class \mathcal{F}_K of all K-canonical f -matchings.*

In the sequel we develop the concept of H -alternating paths and prove the Theorems 2.6–2.10. At the end of this paragraph these results are combined to prove Theorem 2.1.

The theory of integer matchings in graphs is based on the concept of alternating paths. We adapt this concept for our purpose as follows:

Definition 2.2. Let X be a graph and H a subgraph of X . A path W (a cycle C) is called *alternating with respect to H* (H -alternating for short) if and only if for each pair of adjacent edges of W (of C), exactly one belongs to H . A vertex $u \in X$ is said to be an H -alternating path of length 0. Evidently, a single edge e is an H -alternating path of length 1, whether or not $e \in H$.

Lemma 2.3. *Let $F = F(z)$ be a maximum basic f -matching and U the set of the vertices of X which are not saturated by components of F . Then we have:*

- (i) *If C is a cycle of F and if $u \in U$, then there is no $L(F)$ -alternating path between u and a vertex u' of C .*
- (ii) *Let C and C' be two different cycles of F . If F is C -canonical, then there is no $L(F)$ -alternating path between a vertex u of C and a vertex u' of C' .*
- (iii) *Let C be a cycle of F . If F is L-canonical, then C is a chordless cycle of X .*
- (iv) *F is a maximum basic f -matching with respect to each subgraph Y of X satisfying $F \subset Y \subset X$.*

Proof. (i) Take a maximum integer matching M of C which leaves the vertex u' unsaturated and let W be an $L(F)$ -alternating path between u and u' . Define

$$\bar{z}_e = \begin{cases} 1 - z_e & \text{if } e \in W, \\ 1 & \text{if } e \in M, \\ 0 & \text{if } e \in C - M, \\ z_e & \text{otherwise.} \end{cases}$$

Then \bar{z} is feasible and $v(\bar{z}) = v(z) + \frac{1}{2}$ contradicting the optimality of z .

(ii) Analogously, taking integer matchings M and M' of C and C' , respectively, leaving unsaturated just the vertices u and u' , define

$$\bar{z}_e = \begin{cases} 1 - z_e & \text{if } e \in W, \\ 1 & \text{if } e \in M \cup M', \\ 0 & \text{if } e \in C - M \cup C' - M, \\ z_e & \text{otherwise.} \end{cases}$$

Then \bar{z} is feasible and we have $v(\bar{z}) = v(z)$, but $|C(F(\bar{z}))| < |C(F(z))|$ which contradicts the fact that $|C(F(z))|$ is minimum.

(iii) Let u, u' be vertices of C and assume that $\bar{e} = \langle u, u' \rangle$ is a chord of C in X . u and u' divide C into two parts C' and C'' of even and odd length, respectively. Take an integer matching M'' of C'' leaving unsaturated just the vertices u and u' . Define

$$\bar{z}_e = \begin{cases} \frac{1}{2} & \text{if } e = \bar{e}, \\ 1 & \text{if } e \in M'', \\ 0 & \text{if } e \in C'' - M'', \\ z_e & \text{otherwise.} \end{cases}$$

Then \bar{z} is feasible and $v(\bar{z}) = v(z)$, but $|L(F(\bar{z}))| > |L(F(z))|$. This is a contradiction to the maximality of $|L(F)|$.

(iv) Evident.

The next two definitions are given in order to simplify the proof of Theorem 2.6.

Definition 2.4. Let C_1, C_2 be two cycles of a basic f -matching F of X . A path $W = (x_0, x_1, \dots, x_k, x_{k+1})$ is called a *direct path* from C_1 to C_2 with respect to F if and only if

- (1) x_0 belongs to C_1 and x_{k+1} belongs to C_2 .
- (2) All vertices x_1, \dots, x_k are saturated with respect to F by elements of $L(F)$.

Definition 2.5. Let $W = (x_0, x_1, \dots, x_k, x_{k+1})$ be a direct path of X with respect to F connecting two different cycles C_1 and C_2 of F and let s be the number of vertices in $\{x_1, \dots, x_k\}$ which are saturated by an edge of $L(F)$ not belonging to W . Then s is called the *order* of W and is denoted by $\text{ord}(W)$. Evidently, if $\text{ord}(W) = 0$, then W is an $L(F)$ -alternating path connecting two cycles of F .

Now we turn to the case where X possesses an F -factor. In order to compare

different F -factors F^1 and F^2 of a graph X it is very useful to consider the union $F^1 \cup F^2$. Single edges and odd cycles are components of both F^1 and F^2 if and only if they are components of $F^1 \cup F^2$.

Of course, those parts of F^1 and F^2 where F^1 and F^2 differ are of special interest. Therefore, let us consider those components of $F^1 \cup F^2$ that contain vertices which are saturated with respect to F^1 and F^2 by different elements. Since a maximum f -matching F induces a maximum f -matching in each component of X , without loss of generality we may concentrate on components of $F^1 \cup F^2$.

Theorem 2.6. *Let F^1 and F^2 be C -canonical F -factors and let K be a connected component of $F^1 \cup F^2$. Then $F^1 \cap K$ contains at most one cycle and the same holds for $F^2 \cap K$.*

Proof. Suppose that $F^1 \cap K$ contains more than one cycle. Since K is connected, there are direct paths (with respect to F^1) in K between different cycles C_1 and C_2 of $F^1 \cap K$.

Let $W = (x_0, x_1, \dots, x_k, x_{k+1})$ be such a path. By Lemma 2.3(ii), there is no $L(F^1)$ -alternating path between C_1 and C_2 . Thus we have $\text{ord}(W) > 0$. Now, let x_s be a vertex which is saturated with respect to F^1 by an edge $\langle x_s, y \rangle$ of $L(F^1)$ which does not belong to W . Thus both edges $\langle x_{s-1}, x_s \rangle$ and $\langle x_s, x_{s+1} \rangle$ belong to F^2 and therefore x_s lies on a cycle C_3 of F^2 . But this cycle must be the only one of F^2 having vertices common with the path W . Otherwise some part of W would be a direct path of order 0 with respect to F^2 connecting two different cycles of F^2 . Again, this is excluded by Lemma 2.3(ii).

Now let us construct a maximum integer matching $M(F^1)$ on X as it was defined in (1). We can do this in such a way that the vertices x_0 and x_{k+1} remain unsaturated with respect to $M(F^1)$. Next we consider that component K' of $M(F^1) \cup F^2$ which contains the cycle C_3 . Since the edge set of the path W is a subset of $L(F^1) \cup F^2$ and $L(F^1) \subset M(F^1)$ by construction, K' contains both vertices x_0 and x_{k+1} . But K' cannot contain a cycle C_4 of F^2 different from C_3 . Otherwise C_3 and some cycle C_4 of F^2 would be connected by a direct path of order 0 with respect to F^2 . This again is excluded by Lemma 2.3(ii).

By Lemma 2.3(iv), F^2 is an F -factor of $M(F^1) \cup F^2$. Hence, $F^2 \cap K'$ is an F -factor of K' . Let $M(F^2 \cap K')$ be the integer matching of K' as defined in (1). Since K' contains only one cycle of F^2 , namely C_3 , there remains exactly one vertex of K' unsaturated with respect to $M(F^2 \cap K')$, and this leads to a contradiction since $M(F^1)$ is a maximum integer matching on X and therefore on $M(F^1) \cup F^2$, and consequently $M(F^1) \cap K'$ is a maximum integer matching on the component K' of $M(F^1) \cup F^2$, which leaves at least two vertices, x_0 and x_{k+1} , unsaturated.

Thus the assumption that F^1 contains more than one cycle leads to a contradiction establishing the result.

Theorem 2.7. *Let F^1 and F^2 be L-canonical F-factors of X and let K be a component of $F^1 \cup F^2$. If $F^1 \cap K$ contains a cycle of length s , then $F^2 \cap K$ contains a cycle of length s .*

Proof. If $F^1 \cap K$ contains a cycle, then by (2) and by Theorem 2.6 the number of vertices of K is odd. If $F^2 \cap K$ contains no cycle, then this number of vertices is even. Hence both $F^1 \cap K$ and $F^2 \cap K$ contain exactly one cycle. Since $|L(F^1 \cap K)| = |L(F^2 \cap K)|$, both cycles have the same length.

Theorem 2.8. *Let F^1 and F^2 be L-canonical F-factors of X . Then the characteristic vectors of F^1 and F^2 are equal.*

Proof. Let K_1, \dots, K_p be the components of $F^1 \cup F^2$. Then for all $i = 1, 2, \dots, p$ and $j = 1, 2$ the subgraph $K_i \cap F^j$ is an L-canonical F-factor of K_i . By Theorems 2.6, 2.7, $K_i \cap F^1$ and $K_i \cap F^2$ have the same characteristic vector, $i = 1, 2, \dots, p$. This proves the theorem.

It follows that every L-canonical F-factor is K-canonical. Thus, finally, if X possesses an F-factor, then we have $\mathcal{F}_K = \mathcal{F}_L$.

While Theorem 2.6 is true for C-canonical F-factors, Theorem 2.7 can only be proved for L-canonical F-factors. A counterexample is given in Fig. 1 where F^1 and F^2 are two different C-canonical F-factors having cycles of different length. F^2 is L-canonical but F^1 is not.

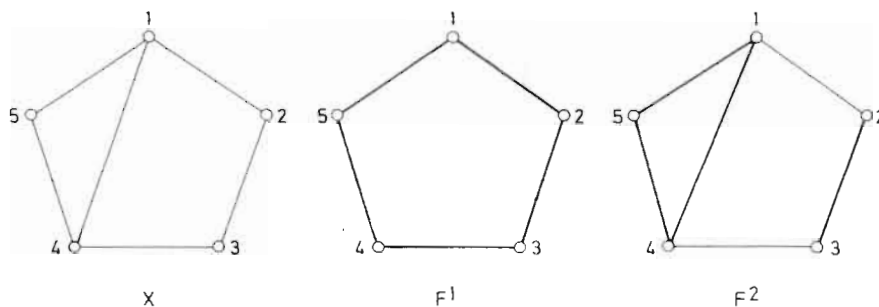


Fig. 1.

Now we turn to the case where X possesses no F-factor and generalize Theorem 2.8 to maximal basic f-matchings. As before let \mathcal{F} be the class of all maximum basic f-matchings and let $F^1 = F(z^{(1)})$ and $F^2 = F(z^{(2)})$ be two different elements of \mathcal{F} . Let $U^j, j = 1, 2$, be the set of vertices which are unsaturated with respect to F^j . Due to Lemma 2.3(iv) we may assume without loss of generality that $X = F^1 \cup F^2$. In this case we have

$$|U^1| = |U^2|, \quad U^1 \cap U^2 = \emptyset$$

and $U^1 \cup U^2$ is a stable set in X . First we prove:

Lemma 2.9. *Let $F^1, F^2 \in \mathcal{F}_L$, $X = F^1 \cup F^2$, $u \in U^1$ and let*

$$W = (u = x_0, x_1, \dots, x_k)$$

be a maximal $L(F^1)$ -alternating path starting in u . Then W has even length and contains at least one edge of $L(F^2)$.

Proof. Evidently, the length of W is at least 1. Since W is maximal, W must be of even length, i.e. $k = 2l$ for some $l \geq 1$.

(a) Suppose $x_k \in U^2$. By Lemma 2.3(i) no vertex of W belongs to a cycle of F^1 or to a cycle of F^2 . Thus at least one edge of W must belong to $L(F^2)$.

(b) Suppose now that $x_k \notin U^2$ and assume that W does not contain an edge of $L(F^2)$. The maximality of W implies that the edge $\langle x_{k-1}, x_k \rangle$ belongs to $L(F^1)$ and that there exists an x_i , $0 \leq i \leq k-2$, such that the edge $\langle x_k, x_i \rangle$ is in F^2 (see Fig. 2).

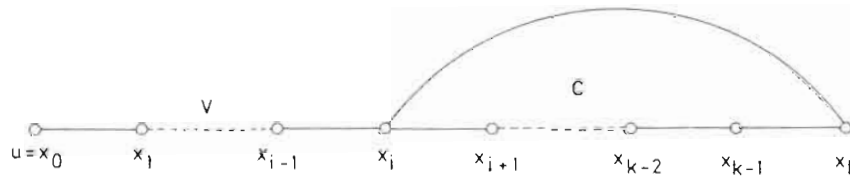


Fig. 2.

If $\langle x_{i-1}, x_i \rangle \in L(F^1)$, then $F^1 = F(z^{(1)})$ is not in \mathcal{F} . For define C to be the cycle $(x_i, x_{i+1}, \dots, x_k, x_i)$ and V to be the path (x_0, \dots, x_i) . Define

$$\bar{z}_e = \begin{cases} \frac{1}{2} & \text{if } e \in C, \\ 1 - z_e^{(1)} & \text{if } e \in V, \\ z_e^{(1)} & \text{otherwise.} \end{cases}$$

Then $v(\bar{z}) = v(z^{(1)}) + \frac{1}{2}$ which contradicts the maximality of F^1 .

From this it follows that $\langle x_i, x_{i+1} \rangle \in L(F^1)$ and $\langle x_{i-1}, x_i \rangle \in F^2$. Hence, x_{i-1}, x_i, x_k are vertices of a cycle D of F^2 . C cannot have a vertex common with a cycle $D' \neq D$ of F^2 . In this case there would be an $L(F^2)$ -alternating path between different cycles of F^2 which is impossible by Lemma 2.3(ii). Furthermore, x_{i+1} is not a vertex of D . Otherwise, $\langle x_i, x_{i+1} \rangle$ would be a cord of D which is impossible by Lemma 2.3(iii). Thus, $\langle x_{i+1}, x_{i+2} \rangle$ must belong to $L(F^2)$, which is a contradiction to our assumption.

The following theorem reduces the general case of arbitrary graphs X to the case where X possesses an F-factor.

Theorem 2.10. Let $F^1 = F(z^{(1)})$ and $F^2 = F(z^{(2)})$ be different L-canonical f -matchings of $X = F^1 \cup F^2$. Then there is a sequence (F_i^1, F_i^2) , $0 \leq i \leq s$, satisfying the following conditions:

- (a) For $0 \leq i \leq s$, F_i^1 and F_i^2 are L-canonical f -matchings of $X_i = F_i^1 \cup F_i^2$.
- (b) $F_0^1 = F^1$, $F_0^2 = F^2$.
- (c) F_s^1, F_s^2 are F -factors in $X_s = F_s^1 \cup F_s^2$.
- (d) $C(F_i^1) = C(F^1)$, $C(F_i^2) = C(F^2)$ for $i = 1, 2, \dots, s$, i.e. the cycle parts never change.
- (e) $|L(F_i^1)| - |L(F_{i+1}^1)| = |L(F_i^2)| - |L(F_{i+1}^2)|$.

Proof. If $U^1 = \emptyset$, then $U^2 = \emptyset$ and the theorem is true with $s = 0$. Suppose therefore that $U^1 \neq \emptyset$ and take some $u \in U^1$. Since $u \notin U^2$, there is an edge $\langle u, u' \rangle \in F^2$. Thus, there is a maximal $L(F^1)$ -alternating path $W = (u = x_0, x_1, \dots, x_k)$ which by Lemma 2.9 is of even length and contains an edge \bar{e} of $L(F^2)$. Define

$$\bar{z}_e = \begin{cases} 1 - z_e^{(1)} & \text{if } e \in W, \\ z_e^{(1)} & \text{otherwise,} \end{cases}$$

$$\begin{aligned} \bar{F}^1 &= F(\bar{z}), & Y &= L(\bar{F}^1) \cap L(F^2), \\ F_1^1 &= \bar{F}^1 - Y, & F_1^2 &= F^2 - Y. \end{aligned}$$

Then we have:

- (i) \bar{F}^1 is L-canonical in X since $|L(\bar{F}^1)| = |L(F^1)|$.
- (ii) $Y \neq \emptyset$ by Lemma 2.9.
- (iii) F_1^1, F_1^2 are L-canonical in $X_1 = F_1^1 \cup F_1^2$ which is strictly contained in X .
- (iv) If $x_k \in U^2$, then $x_k \notin X_1$, hence in this case $|U_1^1| < |U^1|$, otherwise $|U_1^1| = |U^1|$. Starting with F_1^1, F_1^2 and X_1 , instead of F^1, F^2 and X and continuing in this way we get a finite sequence (F_i^1, F_i^2) , where $X_i = F_i^1 \cup F_i^2 \subset X_{i-1}$. Since $C(F^1)$ and $C(F^2)$ never change, the final pair (F_s^1, F_s^2) satisfies (c) and (d).

Finally we combine Theorem 2.8 and Theorem 2.10 to prove Theorem 2.1: Since F_s^1 and F_s^2 are L-canonical in $X_s \subset X$, by Theorem 2.8 they have the same characteristic vector. Hence, by construction, the characteristic vectors of F^1 and F^2 are equal, too.

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